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Fourteen questions from the period 1965–1995

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Abstract

The cited questions are prompted, one each, by results from the doctoral dissertations directed by the author. In the interests of clarification and modernization, the material is recast in the light of recent developments from the literature. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introductory remarks

0.1. Preface

This paper is based on an address given June 17, 1996 to a group of friends and colleagues at the Avila Beach Hotel in Curacao, N.A. There, however, I emphasized the accomplishments in their doctoral dissertations of my 14 Ph.D. students, giving some secondary attention to related unsolved problems; here, in deference to the content of the Questions chosen for exposition and at the suggestion of the editors and referee, the

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celebratory component of those students' work receives only secondary emphasis. There remains, however, enough background material from the theses in question and from the relevant literature to ensure in each case that the question is placed in its historical context; thus the present treatment remains essentially self-contained. It is hardly surprising that with the passage of time (some 15 years, on average), some statements in the original theses have proved susceptible to reworking or improvement or even to correction. In the following sections of the present paper, with the benefit of this hindsight and in an attempt at improvement or simplification, I take the liberty of advancing a definition, theorem, statement or point of view which differs from the original: 2.2, 4.3, 7.7, 7.12, 11.4, 12.2, 13.2.

For purposes of efficient browsing, the following informal Index may be helpful.

0.2. Notation and definitions

The notation ZFC + A indicates that the theorem in question is proved in Zermelo–Fraenkel set theory with Choice, augmented with some additional axiom(s) A. In some cases we specify A precisely. In others, especially when the point is to ask whether the theorem can be proved in ZFC alone, we omit the specifics of A.

When B has been defined or is a familiar construct, the notation $A := B$ defines A to be equal to B.

For S a set and κ an infinite cardinal, we write $[S]^\kappa = \{A \subseteq S\colon |A| = \kappa\}$; the symbols $[S]^{\leqslant\kappa}$ and $[S]^{<\kappa}$ are defined similarly.

Although many of the results given here hold for a large class of spaces, in the interest of simplicity we assume throughout that all (hypothesized) spaces are Tychonoff spaces, i.e., completely regular, Hausdorff spaces. We write $X \simeq Y$ to indicate that X and Y are homeomorphic. For X a space and $x \in X$, the symbol $\mathcal{N}(x)$ denotes the neighborhood system of X at x.

Given a continuous function $f : X \to K$ with K compact, we use the notation $\overline{f}$ to denote the (continuous) Stone extension of f from $\beta(X)$ into K.

For spaces X and Y the notations $C(X)$, $C^*(X)$ and $C(X, Y)$ are used as by Gillman and Jerison [44] and Engelking [34]. For $f \in C^*(X)$, the *cozero-set* and the *zero-set* $\mathrm{coz}(f)$ and $Z(f)$ of f are defined, respectively, to be

$$\{x \in X\colon\ f(x) \neq 0\} \quad \text{and} \quad \{x \in X\colon\ f(x) = 0\};$$

we write $\mathcal{Z}(X) = \{Z(f)\colon\ f \in C(X)\}$.

Given a set $\{X_i\colon\ i \in I\}$ of spaces, we write $X_I := \prod_{i\in I} X_i$; except where specified otherwise, X_I carries the usual product topology. In that context, a *basic set* is an (open) set of the form $U = \prod_{i\in I} U_i$ with each U_i open in X_i for which the *restriction set* $r(U)$ of U, defined by $r(U) = \{i \in I\colon\ U_i \neq X\}$, satisfies $|r(U)| < \omega$.

The discrete space of cardinality α is denoted simply α, while $\langle\alpha\rangle$ denotes the same set in its order topology. For spaces X with Stone–Čech compactification $\beta(X)$ we write $X^* = \beta(X) \setminus X$; in particular, ω^* is the Stone–Čech remainder $\beta(\omega) \setminus \omega$. An ultrafilter p over the (discrete) space α is said to be *uniform* if $\|p\| := \min\{|A|\colon\ A \in p\} = \alpha$. The set of uniform ultrafilters over α is denoted $U(\alpha)$; we write $N(\alpha) = \beta(\alpha) \setminus \alpha$ (the set of *non-uniform* ultrafilters).

1. Stelios A. Negrepontis

En route to developing a homology theory for realcompact spaces, Negrepontis in his thesis [81] determined several facts about the relation of a realcompact space to any space containing it densely. The following definition was useful.

Definition 1.1 (Negrepontis [81]; see also Mrowka [80]). X is *r-embedded* in $Y \supseteq X$ if for $p \in Y \setminus X$ there exists $Z \in \mathcal{Z}(Y)$ such that $p \in Z \subseteq Y \setminus X$. An *r-compactification* of X is a compactification BX of X in which X is (dense and) r-embedded.

It is not difficult to see that for each space X, the following conditions are equivalent:

(i) X is realcompact;

(ii) βX is an r-compactification of X; and

(iii) X admits an r-compactification.

It then follows that every r-embedded subspace A of a realcompact space X is realcompact. (Proof: $\overline{A}^{\beta X}$ is an r-compactification of A.)

Notation 1.2. Given X, the σ-algebra (of subsets of X) generated by $\mathcal{Z}(X)$ is denoted $\mathcal{B}(X)$ and is called the algebra of *Baire* sets of X.

With the preceding as preamble, we reach this result.

Theorem 1.3 (Negrepontis [81, 1.17]). *Every Baire set in a realcompact space is realcompact.*

Proof. It is clear that for any space X, the union of countably many r-embedded subsets of X, and the intersection of arbitrarily many r-embedded subsets of X, is again r-embedded in X. The zero-sets of X and the cozero-sets of X are (evidently) r-embedded in X, so the statement is immediate. □

The same argument, *mutatis mutandis*, shows that a Baire set in a topologically complete space is topologically complete. We turn now to closely related properties.

Theorem 1.4 ([21, 11.8]).
(a) *There are a paracompact space X and $A \in \mathcal{B}(X)$ such that A is not paracompact.*
(b) *Assume* CH. *There is a Lindelöf space X and $A \in \mathcal{B}(X)$ such that A is not Lindelöf.*

Proof. It is a beautiful and well known result of Michael [74] that there is a paracompact space S, which under CH may be chosen to be Lindelöf, such that (with $\mathbb{J}$ denoting the set of irrational numbers in its usual topology) the space $A := S \times \mathbb{J}$ is not normal. Then with $X := S \times [0,1]$ the space $A \in \mathcal{B}(X)$ is as required. □

Corollary 1.5 (Michael [76]).
(a) *There is a sequence $\{X_n\colon n < \omega\}$ of spaces such that each product $\prod_{n<m} X_n$ is paracompact, and $\prod_{n<\omega} X_n$ is not normal.*
(b) *Assume* CH. *There is a sequence $\{X_n\colon n < \omega\}$ of spaces such that each product $\prod_{n<m} X_n$ is Lindelöf, and $\prod_{n<\omega} X_n$ is not normal.*

Discussion 1.6. The question whether the conclusion of Corollary 1.5(b) can be achieved in ZFC alone, without additional axioms, was raised by Michael [75,76] and answered affirmatively by Przymusiński [91]. Later Lawrence [66] found in ZFC a Lindelöf space X and a separable metric space Y such that $X \times Y^m$ is Lindelöf for each $m < \omega$, while $X \times Y^\omega$ is not normal. The question whether such X exists with Y the countable discrete space ω, known as "Michael's problem", remains unsolved in ZFC; such X is sometimes called a *Michael space*. For some conditions necessary for the existence of a Michael space, and for references to relevant work of K. Alster and of D.K. Burke and S. Davis, see Lawrence [65]. At this writing the best results (i.e., those with hypotheses of the form ZFC + A with A minimal) giving the existence of a Michael space are due to J.T. Moore [78].

Since for every Lindelöf space X the spaces $X \times \omega^m$ and $X \times [0,1]$ are Lindelöf, while

$$X \times \omega^\omega \simeq X \times \mathbb{J} \in \mathcal{B}\bigl(X \times [0,1]\bigr),$$

the following question from [21] surfaces as a tame, but still unsolved, version of Michael's problem.

Question 1 (Comfort and Negrepontis [21]). Can the conclusion of Theorem 1.4(b) be established in ZFC?

2. Norman L. Noble

Lemma 2.1 (Noble [82], [83, §4.1]). *Let $\kappa \geqslant \omega$, and let $\mathbb{P}$ and $\mathbb{Q}$ be topological properties such that* (a) *every non-empty subproduct of a product in $\mathbb{P}$ is again in $\mathbb{P}$, and* (b) *for every non-empty product $X_I \in \mathbb{P}$ there is $J \subseteq I$ such that $|I \setminus J| \leqslant \kappa$ and each X_i with $i \in J$ satisfies $X_i \in \mathbb{Q}$. Then for every non-empty product $X_I \in \mathbb{P}$ there is $J' \subseteq I$ such that $|I \setminus J'| \leqslant \kappa$ and each product X_K with $K \in [J']^{\leqslant\kappa}$ satisfies $X_K \in \mathbb{Q}$.*

Proof. For every pairwise disjoint family $\{J_\xi\colon \xi < \kappa^+\}$ of κ^+-many elements of $[I]^{\leqslant\kappa}$ we have $\prod_{\xi<\kappa^+} X_{J_\xi} \in \mathbb{P}$, so some $\xi < \kappa^+$ satisfies $X_{J_\xi} \in \mathbb{Q}$. An easy induction then shows that there is $J'' \in [I]^{\leqslant\kappa}$ such that every $K \in [I]^{\leqslant\kappa}$ with $X_K \notin \mathbb{Q}$ meets J''. Now let $J \subseteq I$ be such that $|I \setminus J| \leqslant \kappa$ and each X_i with $i \in J$ satisfies $X_i \in \mathbb{Q}$, and define $J' = J \setminus J''$. Then J' is as required. □

It is a useful result of A.H. Stone [103], reproved by Ross and Stone [95], that if a non-empty product space X_I is normal, then all but at most countably many of the factors X_i are countably compact. Noble's lemma above permits a stronger conclusion, as follows.

Theorem 2.2 (Noble [82], [83, 4.2]). *For every non-empty, normal space X of the form $X = X_I = \prod_{i\in I} X_i$ there is $J \subset I$ such that $|I \setminus J| \leqslant \omega$ and X_J is countably compact.*

Proof. According to the theorem cited from [103,95], there is $J \subset I$ such that $|I \setminus J| \leqslant \omega$ and each X_K with $K \in [J]^{\leqslant\omega}$ is countably compact (and hence pseudocompact). Since a product space in which each countable subproduct is pseudocompact is itself pseudocompact [46], the space X_I is pseudocompact and hence (being normal) countably compact, as required. □

Discussion 2.3. Noble [83] was drawn to the considerations discussed above by his interest in k-spaces and their products. The argument just given shows in effect that if a product space X_I is a k-space then there is $J \subseteq I$ such that $|I \setminus J| \leqslant \omega$ and X_J is pseudocompact. This conclusion is strengthened in [84]: For every cardinal $\kappa \geqslant \omega$ there is $J' \subseteq I$ such that $|I \setminus J'| \leqslant \kappa$ and each product X_K with $K \in [J']^{\leqslant\kappa}$ is initially κ-compact in the sense that every open cover of cardinality at most κ admits a finite subcover. Noble [83] notes also that a non-empty product space X_I is a k-space if and only if the product of its non-compact factors is a k-space, and he shows [84] that for regular cardinals α the well-ordered space $X = \langle\alpha\rangle$ has X^γ a k-space if and only if $\gamma \leqslant \alpha$, but to the best of my knowledge the following question has not been successfully addressed in the literature.

Question 2. Given $\alpha \geqslant \omega$, what are the spaces X such that X^α is a k-space?

3. Neil Hindman

Definition 3.1 (Gillman–Henriksen). X is an *F'-space* if disjoint cozero-sets in X have disjoint closures.

Definition 3.2. X is an F'_α*-space* ($\alpha > 0$) if every family $\{U_\xi\colon \xi < \alpha\}$ of pairwise disjoint cozero-sets of X satisfies $\bigcap_{\xi<\alpha} \overline{U_\xi} = \emptyset$.

[So $F' = F'_2$, and for $0 < \alpha < \beta$ every F'_α-space is an F'_β-space.]

Discussion 3.3. It is known that in ω^*, disjoint σ-compact sets are separated by a partition. In particular, disjoint cozero-sets are so separated, so ω^* is an F'-space, i.e., no point is in the closure of two disjoint cozero-sets. Could one find in ω^* a point which is in the closure of two disjoint open sets? The answer is "Yes": Even in 1960, when [44] was published, Gillman and Jerison could assert that the fact that ω^* is not extremally disconnected was well known (see [44, p. 270]). Could "two" be replaced by "three" or by higher cardinals? Recalling Walter Rudin's theorem [96] that under CH there are P-points in ω^*, Pierce [90, 21.3] noted (using CH again) that any such point p has a local clopen basis $\{U_\xi\colon \xi < \mathfrak{c}\}$ which is anti-isomorphic (as an ordered set) to the ordinal number $\mathfrak{c}$. Writing each $\xi < \mathfrak{c}$ in the form $\xi = \lambda(\xi) + n(\xi)$ with $\lambda(\xi)$ a limit ordinal and $n(\xi)$ a finite ordinal, we see with

$$W_n := \bigcup \bigl\{U_\xi \setminus U_{\xi+1}\colon n(\xi) = n\bigr\}$$

that $\{W_n\colon n < \omega\}$ is a sequence of pairwise disjoint open subsets of ω^*, with $p \in \overline{W_n}$ for each $n < \omega$. Pierce [90, 25(6)] went on to ask whether his result could be proved without CH. Hindman gave a strong answer to this question, as follows. (Here as usual for X a space and $p \in X$ and $\gamma \geqslant \omega$, we say that p is a γ*-point* of X if there is a family $\mathcal{A}$ of pairwise disjoint open subsets of X with $|\mathcal{A}| = \gamma$ such that $p \in \overline{A}^X$ for each $A \in \mathcal{A}$.)

Theorem 3.4 (Hindman [59, 3.2.4, 3.3.2], [60]).

(a) *There is a* $\mathfrak{c}$*-point in* ω^*; *and*

(b) *assuming CH, every point of* ω^* *is a* $\mathfrak{c}$*-point.*

Hindman's proof left unanswered the question whether the conclusion of Theorem 3.4(b) could be established in ZFC. To review a bit of the subsequent literature, let us use the notation $C(\kappa,\gamma)$ to abbreviate the statement that every uniform ultrafilter p over the (discrete) space κ is a γ-point of $U(\kappa)$. It is not difficult to see that $C(\kappa,\gamma)$ is equivalent to the existence, for each $p \in U(\kappa)$, of $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ such that simultaneously

(i) $|\mathcal{A}| = \gamma$,

(ii) $|A \cap B| < \kappa$ for distinct $A,\ B \in \mathcal{A}$, and

(iii) $|A \cap X| = \kappa$ for all $A \in \mathcal{A}$, $X \in p$.

Statement $C(\omega,\mathfrak{c})$ was proved in ZFC by Balcar and Vojtáš [4]. The attractive conjecture that $C(\kappa, 2^\kappa)$ is a theorem of ZFC for all $\kappa \geqslant \omega$ is thwarted by Baumgartner's model [8] in which, for some $\kappa > \omega$ and for $\gamma = 2^\kappa$, no $\mathcal{A} \subset \mathcal{P}(\kappa)$ satisfies (i) and (ii). But Balcar and Simon [3] proved $C(\kappa, \lambda(\kappa))$ for all regular $\kappa > \omega$; here $\lambda(\kappa)$ is defined by

$$\lambda(\kappa) = \inf\bigl\{|H|\colon\ H \subseteq \kappa^\kappa,\ H \text{ is } \preccurlyeq\text{-unbounded}\bigr\}$$

with $\preccurlyeq$ the partial order on κ^κ given by

$$f \preccurlyeq g \quad \text{if} \quad \bigl|\bigl\{\xi < \kappa\colon\ f(\xi) > g(\xi)\bigr\}\bigr| < \kappa.$$

A consequence of this result is $C(\kappa,\kappa^+)$ for all regular $\kappa > \omega$. The same conclusion for singular κ with $\mathrm{cf}(\kappa) = \omega$ in announced in [3] and proved in [102].

It appears that the outstanding unsolved problems in this corner of mathematics are the following.

Questions 3.
(a) (Balcar and Simon [3]). Does $C(\kappa,\kappa^+)$ hold for singular κ with $\mathrm{cf}(\kappa) > \omega$?
(b) (Comfort and Hindman [15]). Suppose that there is on κ a κ-almost disjoint family of cardinality γ. Does $C(\kappa,\gamma)$ hold?
(c) (Hechler [51]). Is every nowhere dense subset F of ω^* a $\mathfrak{c}$-set—that is, must there exist a family $\mathcal{A}$ of pairwise disjoint open subsets of ω^* with $|\mathcal{A}| = \mathfrak{c}$ such that $F \subseteq \overline{A}^{\omega^*}$ for each $A \in \mathcal{A}$?

4. Milton Don Ulmer

There is an extensive literature, surveyed in some detail in [22], to the effect that under suitable circumstances certain (dense) subspaces S of a product space $X_I = \prod_{i\in I} X_i$ are C^*-embedded or C-embedded in X. Many of these results depend on the fact that functions in (say) $C^*(S)$ or in $C^*(X)$ depend on countably many (or, on some fixed small number) of coordinates. Here is another conclusion of the same flavor, due to Ulmer, based on a very different kind of argument (see also however Kister [64]). Proofs of Theorem 4.1 are available in [108] and in [22, 10.17].

As usual, given $p \in X_I = \prod_{i\in I} X_i$ we write

$$\Sigma = \Sigma(p) = \bigl\{x \in X_I\colon \bigl|\{i \in I\colon x_i \neq p_i\}\bigr| \leqslant \omega\bigr\};$$

the set $\Sigma(p)$ is called the *Σ-product in X based at $p \in X$*.

Theorem 4.1 (Ulmer [108, 4.14(ii)]). *If $\{X_i\colon i \in I\}$ is a set of first-countable spaces and $p \in X_I$, then $\Sigma = \Sigma(p)$ is C-embedded in X_I.*

Theorem 4.1 has the following curious consequence.

Corollary 4.2 (Ulmer [108, 4.16]). *In any product space, a C^*-embedded Σ-product is C-embedded.*

To verify Corollary 4.2 it will be enough to prove the following lemma, which appears to be new; for the application one will take for $\mathcal{U}$ the product topology on the product space $X_I = \prod_{i\in I} X_i$ and for $\mathcal{V}$ the product topology relative to the product $\prod_{i\in I}((X_i)_d)$ with each space $(X_i)_d$ discrete.

Given a space $\langle Y,\mathcal{V}\rangle$ and $A \subseteq Y$, we denote by $A_{\mathcal{V}}$ the set A with the topology inherited from $\langle Y,\mathcal{V}\rangle$.

Lemma 4.3. *Let $X \subseteq Y$ and let $\mathcal{U}$ and $\mathcal{V}$ be topologies on Y such that $\mathcal{U} \subseteq \mathcal{V}$ and X is $\mathcal{V}$-dense in Y. If X is C-embedded in $\langle Y, \mathcal{V}\rangle$ and C^*-embedded in $\langle Y, \mathcal{U}\rangle$ then X is C-embedded in $\langle Y, \mathcal{U}\rangle$.*

Proof. It is enough to show for each $q \in Y \setminus X$ that each $f \in C(X_{\mathcal{U}})$ extends continuously over $(X \cup \{q\})_{\mathcal{U}}$. Since $f \in C(X_{\mathcal{V}})$, there is $\overline{f} \in C((X \cup \{q\})_{\mathcal{V}})$ such that $f \subseteq \overline{f}$, say with $\overline{f}(q) = 0$. The function $g := (f \wedge 1) \vee -1 \in C^*(X_{\mathcal{U}})$ extends to

$$\overline{g} \in C^*\big(\big(X \cup \{q\}\big)_{\mathcal{U}}\big) \subseteq C^*\big(\big(X \cup \{q\}\big)_{\mathcal{V}}\big).$$

Since $\overline{g}|U = \overline{f}|U$ with $U = \overline{g}^{-1}(-1, +1) \in \mathcal{U}|(X \cup \{q\})$ we have $\overline{g}(q) = \overline{f}(q) = 0$ and $\overline{f} \in C((X \cup \{q\})_{\mathcal{U}})$. □

It is worth remarking that although there are many theorems in the literature giving conditions sufficient that a Σ-like subset of a product is C- or C^*-embedded, little is known about (non-obvious) necessary conditions. The thesis of Ulmer [108] contains several examples of Tychonoff spaces X_I some of whose Σ-products are not C^*-embedded. The following question is illustrative, and suggestive of more general questions, concerning the C- and C^*-embedded properties of spaces $Y \subseteq X_I$ such that $\pi_J[Y] = X_J$ for all $J \in [I]^{\leqslant\omega}$ yet Y contains no Σ-product.

Question 4 (Comfort and Negrepontis [22, p. 235]). Let $Y \subseteq X_I = \prod_{i \in I} X_i$ with Y dense. Suppose that $\pi_J[Y]$ is C^*-embedded in X_J for each $J \in [I]^{<|I|}$, and that every $f \in C^*(Y)$ depends on fewer than $|I|$-many coordinates (in the sense that there is $K = K_f \in [I]^{<|I|}$ such that $f(x) = f(y)$ whenever $x, y \in Y$ with $x_K = y_K$). Must Y be C^*-embedded in X?

5. Victor Saks

Discussion 5.1. It was proved in 1966 by Scarborough and Stone [101] that a non-empty product space $X_I = \prod_{i \in I} X_i$ of spaces is countably compact if and only if each subproduct X_J with $J \subseteq I$ and $|J| \leqslant 2^{2^{\mathfrak{c}}}$ is countably compact. Saks and I wondered whether that exponent $2^{2^{\mathfrak{c}}}$ was best possible. In solving that problem, Saks found it convenient to use the following concept due to Bernstein [9] (closely related to the "producing" relation introduced earlier by Frolík [38,39]).

Definition 5.2 (Bernstein [9]). For $p \in \omega^*$, a (Tychonoff) space X is p-compact if every continuous function $f : \omega \to X \subseteq \beta(X)$ satisfies $\overline{f}(p) \in X \subseteq \beta(X)$.

The theorem I want to cite now from Saks' dissertation [97, Theorem 25], as cast more elegantly and expanded in his later paper with Ginsburg [45, 2.6], showed only for a fixed space X that every power X^α is countably compact if and only if $X^{2^{\mathfrak{c}}}$ is countably compact; the trick was to show that the "weaker" of these conditions is equivalent to this: there is $p \in \omega^*$ such that X is p-compact. Later Saks [98, 2.3], [99] and I [13] noticed that the argument from [97,45] adapts with no essential change to families of spaces. Again for simplicity we state the theorem here in the Tychonoff context (and we use the Stone–Čech compactification in the proof). For details of several more general theorems see Saks [98, 99].

Theorem 5.3 (Saks [98,99]; see also [13]). *If $X_I = \prod_{i \in I} X_i$ and each $J \subseteq I$ with $|J| \leqslant 2^{\mathfrak{c}}$ has X_J countably compact, then X_I is countably compact.*

Questions 5.

(a) Is there a space X such that $X^{2^{\mathfrak{c}}}$ is not countably compact, but X^{α} is countably compact for each $\alpha < 2^{\mathfrak{c}}$?

(b) Is there a set $\{X_i \colon i \in I\}$ of spaces such that X_I is not countably compact, but for which X_J is countably compact for each $J \in [I]^{<2^{\mathfrak{c}}}$? Can one arrange that X_J is countably compact for every proper subset J of I?

Questions 5 should be taken in ZFC. Indeed Saks [98, Section 2], [99] has answered some of these questions affirmatively in certain extended axiom systems of the form ZFC + A. Closely related results are given by Yang [112].

6. Liam O'Callaghan

Discussion 6.1. It is not difficult to find, in response to an informal inquiry from Ronnie Levy, spaces Y such that $Y \simeq Y^*$. Indeed let X be an extremally disconnected space which is nowhere locally compact, so that $\beta(X^*) = \beta(X)$ (see [44, 6M.2]), let X_i $(i = 0, 1)$ be disjoint copies of X, and let Y be the "disjoint union" $Y = X_0 \cup_d X_1^*$ of X_0 with the Stone–Čech remainder X_1^* of X_1. Then $Y^* = X_0^* \cup_d X_1 \simeq Y$, as required. It is more challenging (and often impossible), when a space X is given in advance, to find Y such that $X \subseteq Y \subset \beta(X)$ and $Y \simeq Y^*$. The following results illuminate this situation.

Lemma 6.2. *Let X be a space, let $p \in X$ with $\chi(p, X) \leqslant \omega$, and let $f \in C(X, \beta(X))$. Then $f(p) \in \upsilon(X)$ or f is constant on some neighborhood of p.*

Lemma 6.3. *Let X be a locally compact space with the property that there exist a homeomorphism h from X into X^* and disjoint open-and-closed subspaces X_i of X $(i = 0, 1)$ such that $X = X_0 \cup X_1$, $h[X]$ is C^*-embedded in X^*, and $h[X_i] \subseteq X^*_{1-i}$ $(i = 0, 1)$. Then there is a space Y such that $X \subseteq Y \subseteq \beta(X)$ and $Y \simeq Y^*$.*

Theorem 6.4 (O'Callaghan [86]; see also [23, 2.6]).

(1) *Let X be first countable and realcompact and suppose that there is Y such that $X \subseteq Y \subseteq \beta(X)$ and $Y \simeq Y^*$. Then X is discrete and Y is pseudocompact.*

(2) *For every infinite, discrete space X there is a pseudocompact space Y such that $X \subseteq Y \subseteq \beta(X)$ and $Y \simeq Y^*$.*

The foregoing results suggest these questions.

Questions 6.

(a) Characterize those Tychonoff spaces X for which $X \simeq X^*$, and those for which some Y such that $X \subseteq Y \subseteq \beta(X)$ satisfies $Y \simeq Y^*$.

(b) Characterize those Tychonoff spaces X for which some compactification $B(X)$ of X satisfies $X \simeq B(X) \setminus X$, and those for which some compactification $B(X)$ and some space Y with $X \subseteq Y \subseteq B(X) = B(Y)$ satisfy $Y \simeq B(Y) \setminus Y$.

7. Teklehaimanot Retta

Notation 7.1. For $\alpha \geqslant \omega$ and X a space, a G_α*-subset* of X is an intersection of fewer than α-many open subsets of X.

Thus a set is a G_ω-subset if and only if it is open, and the G_{ω^+}-subsets of a space X are what we usually call the G_δ-subsets of X.

Definition 7.2. Let $\alpha \geqslant \omega$ and let $X = \langle X, \mathcal{T} \rangle$ be a Tychonoff space. Then X_α denotes the set X with the smallest topology $\mathcal{T}(\alpha)$ such that

(i) $\mathcal{T}(\alpha) \supseteq \mathcal{T}$, and

(ii) every G_α-subset of X is open in X_α.

Given $\alpha \geqslant \omega$ and $\langle X, \mathcal{T} \rangle$, we use the three notations X_α, $\langle X, \mathcal{T}(\alpha) \rangle$ and $\langle X, \mathcal{T} \rangle_\alpha$ interchangeably.

Having agreed to restrict attention to Tychonoff spaces, we should remark that X_α is such whenever X is such.

Definition 7.3. Let $\alpha \geqslant \omega$. A Tychonoff space X is α*-compact* if X is closed in $(\beta X)_\alpha$.

Remarks 7.4.

(a) The (Tychonoff) ω-compact spaces are exactly the compact spaces.

(b) The ω^+-compact Tychonoff spaces X are exactly those such that for every $p \in X^*$ there is continuous $f : \beta X \to \mathbb{R}$ such that $f(p) = 0$ and $f > 0$ on X. These are the *realcompact* spaces introduced by Hewitt [55]. See [44] for an extended study of these spaces.

(c) If $\omega \leqslant \gamma \leqslant \alpha$ and X is a γ-compact Tychonoff space then X is α-compact.

Retta [94] gives clearly his own formulation of several properties equivalent to that of Definition 7.3. The concept had been introduced (in other language) by Herrlich [52,53] and studied intensively by Hong [61] and Bhaumik [10], among others. The following characterization is useful.

Theorem 7.5. *Let $\alpha \geqslant \omega$ and let X be a Tychonoff space. Then X is α-compact if and only if there is a Hausdorff compactification BX of X such that X is closed in $(BX)_\alpha$.*

Discussion 7.6. It is not difficult to prove, as in Herrlich [52], Hong [61], and Bhaumik [10], that for $\alpha \geqslant \omega$ the property α-compactness is a *universal topological property* in the sense that

(a) every compact Hausdorff space is α-compact,
(b) every closed subset of an α-compact space is α-compact, and
(c) the product of any set of α-compact spaces is α-compact.

Theorem 7.7 (Retta [94, 3.18]; see also [24, 4.6]). *Let $\alpha \geqslant \omega$ and let $\langle X, \mathcal{T}\rangle$ be an α-compact Tychonoff space. If $\mathcal{U}$ is a Tychonoff topology for X such that $\mathcal{T} \subseteq \mathcal{U} \subseteq \mathcal{T}(\alpha)$, then $\langle X, \mathcal{U}\rangle$ is an α-compact space.*

Proof. To see that X is closed in $(\beta(X,\mathcal{U}))_\alpha$ we show that every $p \in \overline{X}^{(\beta(X,\mathcal{U}))_\alpha}$ satisfies $p \in X$. The (continuous) identity function

$$\mathrm{id}: \langle X, \mathcal{U}\rangle \twoheadrightarrow \langle X, \mathcal{T}\rangle$$

extends continuously to

$$\overline{\mathrm{id}}: \beta(X,\mathcal{U}) \twoheadrightarrow \beta(X,\mathcal{T}),$$

and the function $\overline{\mathrm{id}}$ remains continuous when viewed as a function from $(\beta(X,\mathcal{U}))_\alpha$ onto $(\beta(X,\mathcal{T}))_\alpha$. Thus $\overline{\mathrm{id}}(p) \in \overline{X}^{(\beta(X,\mathcal{T}))_\alpha}$, so $\overline{\mathrm{id}}(p) \in X$. For notational convenience write $\overline{\mathrm{id}}(p) = x \in X$ and (assuming for the moment that the desired conclusion fails, i.e., that $p \notin X$) choose a compact neighborhood K of x in $\beta(X,\mathcal{U})$ such that $p \notin K$. It is clear that $x \notin \overline{X \setminus K}^{(X,\mathcal{U})}$.

From $p \notin K = \overline{K}^{\beta(X,\mathcal{U})}$ follows $p \notin \overline{K}^{(\beta(X,\mathcal{U}))_\alpha}$ and hence $p \in \overline{X \setminus K}^{(\beta(X,\mathcal{U}))_\alpha}$. It follows that $x = \overline{\mathrm{id}}(p) \in \overline{X \setminus K}^{(\beta(X,\mathcal{T}))_\alpha}$ and hence (since $x \in X$ and $X \setminus K \subseteq X$) that $x \in \overline{X \setminus K}^{(X,\mathcal{T}(\alpha))}$. Thus $x \in \overline{X \setminus K}^{(X,\mathcal{U})}$, a contradiction. $\square$

It follows from Theorem 7.7 that if X is a realcompact space then X_{ω^+} is realcompact (a result first stated explicitly in the literature by Frolík [40, Theorem 4]; for citations to work of Misra and Wheeler, and for a derivation due to Anthony W. Hager of this result from early work of Hewitt [56], see [24, 4.8]). So in particular, every realcompact metric space has non-measurable cardinality (cf. [69], [44, 15.24]). Here are two less familiar corollaries.

Corollary 7.8. *Let $\langle X, \mathcal{T}\rangle$ be a realcompact metric space and let $\mathcal{U}$ be a Tychonoff topology for X such that $\mathcal{T} \subseteq \mathcal{U}$. Then $\langle X, \mathcal{U}\rangle$ is realcompact.*

Corollary 7.9. *Let $\omega \leqslant \kappa \leqslant \alpha$, let $\{X_i\colon i \in I\}$ be a set of α-compact Tychonoff spaces, and let $\mathcal{U}$ be the κ-box topology on $X_I := \prod_{i\in I} X_i$. Then $\langle X_I, \mathcal{U}\rangle$ is α-compact.*

Proof. The product topology $\mathcal{T}$ on X_I is α-compact and satisfies $\mathcal{T} \subseteq \mathcal{U} \subseteq \mathcal{T}(\alpha)$, so Theorem 7.6 applies. $\square$

Discussion 7.10. Now for $\alpha \geqslant \omega$ let $\mathfrak{m}(\alpha)$ denote the first measurable cardinal greater than or equal to α. (If no such cardinal $\mathfrak{m}(\alpha)$ exists, the remaining statements of this section will require the obvious appropriate modification.) Retta [94] shows that for a discrete space D the following conditions are equivalent.

(a) D is α-compact;
(b) D is $\mathfrak{m}(\alpha)$-compact; and
(c) $|D| < \mathfrak{m}(\alpha)$.

From this it follows that a Tychonoff space X such that $X = X_{\mathfrak{m}(\alpha)}$ is α-compact if and only if it is $\mathfrak{m}(\alpha)$-compact, so one achieves finally the following corollary to Theorem 7.7.

Corollary 7.11 ([24, 6.1]). *Let $\alpha \geqslant \omega$ and let $\langle X, \mathcal{T} \rangle$ be an α-compact Tychonoff space. Then*

(a) *If $\mathcal{U}$ is a Tychonoff topology for X such that $\mathcal{T} \subseteq \mathcal{U} \subseteq \mathcal{T}(\mathfrak{m}(\alpha))$, then $\langle X, \mathcal{U} \rangle$ is an $\mathfrak{m}(\alpha)$-compact space; and*
(b) *$\langle X, \mathcal{T}(\mathfrak{m}(\alpha)) \rangle$ is α-compact.*

Corollary 7.12. *Let $\alpha \geqslant \omega$ and let X be an α-compact space in which each point is a $G_{\mathfrak{m}(\alpha)}$-point. Then $|X| < \mathfrak{m}(\alpha)$.*

Proof. The discrete space $X_{\mathfrak{m}(\alpha)}$ is α-compact by Theorem 7.9(b), so $|X| = |X_{\mathfrak{m}(\alpha)}| < \mathfrak{m}(\alpha)$. □

Remarks 7.13.

(a) The case $\alpha = \omega$ of Corollary 7.12 reduces to the fact that a compact, discrete space is finite, while the case $\alpha = \omega^+$ of Corollary 7.12 is this statement of Juhász [62]: A realcompact space X in which each point is a $G_{\mathfrak{m}(\omega^+)}$-point satisfies $|X| < \mathfrak{m}(\omega^+)$.
(b) Corollary 7.12, together with other consequences of an argument concerning complete uniform spaces, has been proved by Hager, Reynolds and Rice [48,49]. See also Kato [63] and Williams [111] for related results.

It is not difficult to see, as is noted in [14], that for a given α-compact Tychonoff space $\langle X, \mathcal{T} \rangle$ the set of α-compact Tychonoff topologies on X which contain $\mathcal{T}$ contains a largest member $\mathcal{M} = \mathcal{M}(\mathcal{T})$. (Indeed with

$$\mathfrak{t} = \{\mathcal{U} \supseteq \mathcal{T} \colon \mathcal{U} \text{ is an } \alpha\text{-compact Tychonoff topology for } X\},$$

the topology $\mathcal{M}$ is realized as the topology inherited by the "diagonal copy" of X in the product space $\prod_{\mathcal{U} \in \mathfrak{t}} \langle X, \mathcal{U} \rangle$.) These extremal topologies $\mathcal{M}(\mathcal{T})$ have not been widely studied. Among the questions posed in [14] is this.

Question 7 (Comfort et al. [14]). Let $\alpha \geqslant \omega$ and let $\langle X, \mathcal{T} \rangle$ be an α-compact Tychonoff space. Are the topologies $\mathcal{M}(\mathcal{T})$ and $\mathcal{T}(\mathfrak{m}(\alpha))$ equal? What about the case $\alpha = \omega^+$?

Remark 7.14. Question 7 assumes interest and content only in those models of ZFC in which $\mathfrak{m}(\alpha)$ exists (and satisfies $\mathfrak{m}(\alpha) \geqslant |X|$), for if $\mathfrak{m}(\alpha)$ does not exist or $|X| < \mathfrak{m}(\alpha)$, then the α-compact topologies $\mathcal{M}(\mathcal{T})$ and $\mathcal{T}(\mathfrak{m}(\alpha))$ are both equal to the discrete topology on X.

8. Charles Frank Waiveris, Jr.

Discussion 8.1. A well known construction of Novák [85] and Terasaka [104] shows that the countably infinite discrete space ω can be written in the form $\omega = Y_0 \cap Y_1$ for suitably chosen countably compact subspaces Y_i $(i = 0, 1)$ of $\beta(\omega)$. (The spaces Y_i are then countably compact spaces whose product is not pseudocompact.) Improving this statement, Frolík [36,37] showed that for every space X which is separable metric or discrete there is a family $\{Y_\xi\colon \xi < 2^{\mathfrak{c}}\}$ of $2^{\mathfrak{c}}$-many countably compact spaces such that $X = Y_\xi \cap Y_\eta$ whenever $\xi < \eta < 2^{\mathfrak{c}}$. These results prompted Waiveris [109] to make the following definition and to prove the following results.

Definition 8.2 (Waiveris [109]). A subspace Y of a space S is *extra countably compact* (relative to S) if every infinite subset of S has an accumulation point in Y.

Obviously, an extra countably compact subspace Y of a space S is itself countably compact (in the inherited topology). It has been noted in conversation by Melvin Henriksen that a space X is an intersection of (perhaps many) extra countably compact subsets of $\beta(X)$ if and only if no non-trivial sequence in $\beta(X)$ converges to a point of X^*.

Theorem 8.3 (Waiveris [109]; see also [26]).
- (a) *If X is realcompact or an F-space, then there is a family $\{Y_\xi\colon \xi < 2^{\mathfrak{c}}\}$ of $2^{\mathfrak{c}}$-many extra countably compact spaces such that $X = Y_\xi \cap Y_\eta$ whenever $\xi < \eta < 2^{\mathfrak{c}}$; and*
- (b) *if X is topologically complete then there is a family $\{Y_\xi\colon \xi < 2^{\mathfrak{c}}\}$ of $2^{\mathfrak{c}}$-many countably compact spaces such that $X = Y_\xi \cap Y_\eta$ whenever $\xi < \eta < 2^{\mathfrak{c}}$.*

Waiveris [109] showed also that if there exists a measurable cardinal, so that not every topologically complete space is realcompact, then there is a topologically complete space (indeed, a metric space) X which is not the intersection of any two extra countably compact subsets of $\beta(X)$.

The work of Waiveris leaves some attractive questions unanswered, some of them solved subsequently by Saks [100]. To state one which remains open, let us for a cardinal number κ denote by $\mathcal{E}(\kappa)$ the class of spaces X which can be written as the intersection of fewer than κ-many extra countably compact subspaces of $\beta(X)$. Then surely $\mathcal{E}(\kappa) \subseteq \mathcal{E}(\lambda)$ when $\kappa < \lambda$.

Questions 8 (Waiveris [109]; see also [26]).
- (a) For what cardinals κ and λ is $\mathcal{E}(\kappa) = \mathcal{E}(\lambda)$?
- (b) For what cardinals κ is $\mathcal{E}(\kappa) = \mathcal{E}(3)$?

9. Thomas Joseph Peters

Discussion 9.1. Let us say here for convenience, slightly altering notation introduced by Fine and Gillman [35], that for X dense in Y a point $p \in Y \setminus X$ is a *remote point* for X (in

Y) if there is no nowhere dense subset D of X such that $p \in \overline{D}^Y$. When X has a remote point in $\beta(X)$, we say simply that X *has a remote point*. It is not difficult to prove [87, 4.1] that if X has a remote point p in some compactification BX of X, then X has a remote point; indeed with $\mathrm{id}: X \twoheadrightarrow X \subseteq BX$ the identity function and $\overline{\mathrm{id}}: \beta(X) \twoheadrightarrow BX$ its Stone extension, every $q \in \overline{\mathrm{id}}^{-1}(\{p\})$ is a remote point for X in $\beta(X)$.

Fine and Gillman [35] showed, assuming CH, that $\mathbb{R}$ has remote points. The same conclusion was achieved without CH, and extended substantially, by Chae and Smith [12] and independently, in a far-reaching paper, by van Douwen [28]. This latter paper shows (Theorem 4.1) that a normal, non-pseudocompact space X with countable π-weight has remote points. The same conclusion was achieved by Chae and Smith [12] for what Peters [87] subsequently called a σ-π-space, i.e., a space with a σ-locally finite π-base. Attention then focused naturally on identifying spaces X which (are normal and non-pseudocompact and which) are σ-π-spaces. Peters [87] showed for (infinite, discrete) cardinals α and γ that α^γ is a σ-π-space if and only if $\alpha \geqslant \gamma$. This yields the following nice contribution to the theory of remote points.

Theorem 9.2 (Peters [87, 4.6]). *The space $\omega_1^{\omega_1}$ has remote points.*

Proof. Since $\omega_1^{\omega_1}$ is a σ-π-space and is dense in $\omega_1^{\omega} \times (\beta(\omega_1))^{\omega_1 \setminus \omega}$, this latter space is a σ-π-space; being non-pseudocompact, and paracompact and hence normal, the space $\omega_1^{\omega} \times (\beta(\omega_1))^{\omega_1 \setminus \omega}$ has remote points by the cited theorem of Chae and Smith. Hence its dense subspace $\omega_1^{\omega_1}$ has remote points. $\square$

Perhaps the most accessible of the many outstanding challenging questions concerning remote points are these. Here as usual we say that a space X is a ccc-*space* if every family of non-empty pairwise disjoint open subsets of X is countable.

Questions 9.
(a) (Peters [87, 4.8]). When $\omega \leqslant \alpha < \gamma$, does the space α^γ have remote points?
(b) (Dow [30]). Does every non-compact ccc-space have remote points?

An affirmative answer to Question 9(a) was given by van Mill [77] for the case $\alpha = \omega$, $\gamma = \omega_1$ and later by Dow [30] for the case $\alpha = \omega$ (with arbitrary $\gamma > \omega$), but to my best knowledge other cases remain open. The reader intrigued by Question 9(b) may consult Dow [29], where it is shown that every non-pseudocompact ccc-space of weight at most $\mathfrak{c}$ has remote points.

There are other questions and examples, sometimes counter-intuitive, concerning the existence of remote points in products. The interested reader might consult Dow and Peters [32], where *inter alia* the following statements are proved.

Theorem 9.3 (Dow and Peters [32]).
(a) [ZFC + A] *For each X, there is a cardinal κ such that $\kappa \times X$ has remote points*;
(b) *for each cardinal κ, there is a space X such that $\kappa \times X$ has no remote points*;

(c) *there are non-pseudocompact spaces X and Y, each with no remote points, such that $X \times Y$ has remote points*; *and*
(d) [ZFC + A] *there are non-pseudocompact spaces X and Y, each having remote points, such that $X \times Y$ has no remote points.*

For other results deriving from and related to the thesis of Peters [87], see [88,89].

10. George Baloglou

Discussion 10.1. For a space X let $\kappa(X)$ denote the compact-covering number of X, that is, the least number of compact sets required to cover X.

It is easy to see, perhaps using the Baire Category Theorem and Baire's theorem asserting that the completely metrizable space ω^ω is homeomorphic to the space $\mathbb{J}$ of irrational numbers in its usual topology, that $\omega < \kappa(\mathbb{J}) \leqslant \mathfrak{c}$. From CH then follows $\kappa(\mathbb{J}) = \omega_1 = \mathfrak{c}$. Here is a remark of a slightly different flavor [50], which is clear from the fact that a compact subspace of $\mathbb{J}$ is nowhere dense in $[0,1]$: if MA + ¬CH is assumed then $\kappa(\mathbb{J}) = \mathfrak{c} > \omega_1$.

There are few genuine surprises in mathematics but I believe that the paragraph above gives rise to a simple example which runs counter to the intuition of any set-theoretic topologist. Consider the following situation. Suppose you are given an infinite cardinal γ and spaces X and Y such that $\kappa(X) < \kappa(Y)$; what can be said about $\kappa(X^\gamma)$ versus $\kappa(Y^\gamma)$? Surely something like 99 topologists out of 100 will answer with assurance that $\kappa(X^\gamma) \leqslant \kappa(Y^\gamma)$. Not so! At least, the suggested implication

$$\kappa(X) < \kappa(Y) \Rightarrow \kappa(X^\gamma) \leqslant \kappa(Y^\gamma)$$

cannot be proved in ZFC. To see this let X and Y denote respectively the discrete space ω and the space $\langle\omega_1\rangle$ of countable ordinals in its usual order topology, take $\gamma = \omega$, and work in the axiom system MA + ¬CH. Then $\kappa(X) = \omega < \omega_1 = \kappa(Y)$ (in ZFC); but as we have seen it follows from MA + ¬CH that

$$\mathfrak{c} = \kappa(\mathbb{J}) = \kappa(X^\omega) > \kappa(Y^\omega) = \omega_1,$$

the last equality deriving from the fact that Y^ω is the union of the ω_1-many compact sets $[0,\xi]^\omega$ ($\xi < \omega_1$).

A little study of that example suggests the following theorem.

Theorem 10.2 (Baloglou [5, 2.3.4]; see also [7, 2.7]). *The following conditions are equivalent.*
(a) *GCH*;
(b) *every cardinal $\alpha \geqslant \omega$ satisfy $\kappa(\langle\alpha^+\rangle^\alpha) = 2^\alpha$*;
(c) *every space X and every cardinal γ satisfies $\kappa(X^\gamma) = (\kappa(X))^\gamma$*;
(d) *every set $\{X_i\colon i \in I\}$ of spaces satisfies $\kappa(X_I) = \prod_{i\in I} \kappa(X_i)$.*

There are in [5,7] several questions about the numbers $\kappa(X)$. To state one which to my best knowledge remains unsolved, let us for spaces X and Y write $X \sim Y$ if $\kappa(X^\beta) = \kappa(Y^\beta)$ for all cardinals β, and for $\alpha \geqslant \omega$ let Λ_α denote the proper class of spaces X such that $\kappa(X) = \alpha$.

Questions 10.

(a) (Baloglou [5]; see also [7]). Into how many equivalence classes does $\sim$ partition the class Λ_α? Is $|\Lambda_\alpha/{\sim}| > 2$ possible (for some α, in some models of ZFC)?

(b) (Baloglou [5, 3.3.6]; see also [7, 4.8ff.]). For $X \in \Lambda_\alpha$ and $\gamma \geqslant \omega$, is $\kappa(\langle\alpha\rangle^\gamma) \leqslant \kappa(X^\gamma)$ a theorem of ZFC?

Discussion 10.3. Concerning Question 10(a) we note that $|\Lambda_\omega/{\sim}| = 1$, while from Theorem 10.2 it follows that GCH yields $|\Lambda_\alpha/{\sim}| = 1$ for all $\alpha \geqslant \omega$. It is noted in [5, 3.1.5] that essentially the argument used above to prove in the system MA + ¬CH that $\kappa(\mathbb{J}) = \mathfrak{c} > \omega_1 = \kappa(\langle\omega_1\rangle^\omega)$ shows (in the same axiom system) that $|\Lambda_\alpha/{\sim}| \geqslant 2$ for every regular α such that $\omega < \alpha < \mathfrak{c}$. The inequality indicated in Question 10(b) is established in [5, 3.3.2], [7, 4.5(c)] for spaces $X \in \Lambda_\alpha$ which are *hemicompact* in the sense that X may be written in the form $X = \bigcup_{\xi<\alpha} K_\xi$ with each K_ξ compact so that every compact $K \subseteq X$ satisfies $K \subseteq \bigcup_{\xi<\eta} K_\xi$ for some $\eta < \alpha$.

For other results related to and extending those of Baloglou's thesis [5], see Baloglou [6].

11. Salvador Garcia-Ferreira

Discussion 11.1. Given ultrafilters p and q over a (discrete) set α, we write $p \leqslant_{RK} q$ (and we say that *p precedes q* in the Rudin–Keisler order) if there is $f : \alpha \to \alpha \subseteq \beta(\alpha)$ such that $\overline{f}(q) = p$. Strictly speaking the relation $\leqslant_{RK}$ is not a partial order but a pre-order on $\beta(\alpha)$, since from $p \leqslant_{RK} q \leqslant_{RK} p$ follows not $p = q$ but only $p \sim q$ in the sense that there is a permutation h of α such that $\overline{h}(p) = q$. In this informal discussion for simplicity we treat $\langle\beta(\alpha), \leqslant_{RK}\rangle$ as a partially ordered set. We say for a cardinal number κ and for $S \subseteq \beta(\alpha)$ that *S is κ-directed* by $\leqslant_{RK}$ if for every $A \in [S]^{<\kappa}$ there is $q \in S$ such that each $p \in A$ satisfies $p \leqslant_{RK} q$.

There is an extensive literature on directedness properties of $\beta(\alpha)$ and its subsets. For example, Blass [11, 5.10] and Comfort and Negrepontis [19, 4.4(a)] showed respectively that $\beta(\alpha)$ is α^+- and $(2^\alpha)^+$-directed. From this latter result it follows that if $(2^\alpha)^+ = 2^{2^\alpha}$ then $\beta(\alpha)$ admits a cofinal well-ordered subset (necessarily of cardinality 2^{2^α}), while in [20, 10.15] one finds a short proof of a converse-like statement due to Shelah: If $(2^\alpha)^+ < 2^{2^\alpha}$, then in $\beta(\alpha)$ every set of cardinality 2^{2^α} contains a subset of the same cardinality whose elements are pairwise $\leqslant_{RK}$-incomparable (so surely no linearly ordered subset of $\beta(\alpha)$ can be cofinal).

Among the many contributions of the thesis of Garcia-Ferreira [41] to directedness properties of $\langle\beta(\alpha), \leqslant_{RK}\rangle$ and its subsets is the following.

Theorem 11.2 (Garcia-Ferreira [41, 3.2.37], [42]). *For $\alpha \geqslant \omega$, the following conditions are equivalent.*

(a) *α is regular*;

(b) *$N(\alpha)$ is p-compact for all $p \in N(\alpha)$*;

(c) *$N(\alpha)$ is α-directed.*

(For a proof of the implication (a) $\Rightarrow$ (b) which is perhaps more straightforward than that given in [41, 3.2.37], [42], the reader might consult [19, §4]; see also [20, 10.9].)

Elsewhere in the thesis [41], Garcia-Ferreira introduces on α^* (in particular, on ω^*) an equivalence relation $\simeq_C$ defined as follows: $p \simeq_C q$ if every p-compact space is q-compact and conversely. The equivalence relation $\simeq_C$ on ω^* is investigated vigorously in [41, Chapter IV], but the following question is left unresolved. (Here as in [41] for $p \in \omega^*$ we denote by $T_C(p)$ the $\simeq_C$-equivalence class of p.)

Question 11 (Garcia-Ferreira [41, 4.1.55], [43, 3.9]). For $p \in \omega^*$, is the space $T_C(p)$ p-compact?

Remark 11.3. The proof given in [41, 4.1.54] and in [43, 3.8] suffices to answer Question 11 affirmatively for P-points $p \in \omega^*$, but the question is open even when $p \in \omega^*$ is a weak-P-point.

12. Francisco Javier Trigos-Arrieta

Discussion 12.1. With every topological group G there is associated a compact Hausdorff group bG—the so-called *Bohr compactification* of G—and a continuous homomorphism b from G onto a dense subgroup of bG. Among such compact groups and continuous homomorphisms, bG and b are determined by this property: for every continuous homomorphism h from G into a compact group K there is a continuous homomorphism $\overline{h}$ from bG into K such that $h = \overline{h} \circ \mathrm{b}$. (See Heyer [57, V§4] for a careful examination of bG and its properties. For an early construction when G is a locally compact Abelian Hausdorff group, see Anzai and Kakutani [2]; for such groups G the group here denoted bG is there called the *universal Bohr compactification* of G. It appears that bG was first defined and examined in the unrestricted setting by Weil [110], then independently by Alfsen and Holm [1]; in the terminology of [110, §12] and [1], the group here denoted bG is called the *groupe compact attaché à G* and the *maximal compact representation* of G, respectively. See also Loomis [68, §41].) Restricting attention now in the interest of simplicity to Abelian topological groups G which are in addition *maximally almost periodic* in the sense that the group $\widehat{G}$ of continuous homomorphisms from G to the circle group $\mathbb{T}$ separates points of G (so that the map $\mathrm{b}: G \to \mathrm{b}G$ is an isomorphism), we denote by G^+ the group G with the topology induced by $\widehat{G}$—that is, with the topology inherited from bG. It is easy to see in this context for such groups G and H that if $h \in \mathrm{Hom}(G, H)$ is continuous, then also $h: G^+ \to H^+$ is continuous. Portions of the thesis of Trigos-

Arrieta [107], and most of what follows below in this section, are devoted to questions in the converse direction.

That thesis [107], which covers much ground, contains several results which generalize this familiar, useful result of Glicksberg [47, 1.2] (cf. also Leptin [67]): For G a locally compact Abelian group, a subset of A of G is compact in the topology inherited from G if and only if A is compact in the topology inherited from G^+. In the terminology of Trigos-Arrieta [107], one says that every locally compact Abelian group *respects compactness*. For a more extensive class of groups which respect compactness, and for examples of maximally almost periodic groups which do not respect compactness, see Remus and Trigos-Arrieta [93]. For generalizations of Glicksberg's theorem, see [25,79].

Glicksberg's theorem has the following unexpected consequence, noted by Glicksberg [47] and proved in the group-theoretic context by Trigos-Arrieta [107, 6.32].

Lemma 12.2 (Glicksberg [47, 2.2]). *Let X be a locally compact space and let $f : X \to H$ with H a locally compact Abelian group. If $f : X \to H^+$ is continuous then $f : X \to H$ is continuous.*

Proof. We show that if x_λ is a net in X and $x_\lambda \to x \in X$, then $f(x_\lambda) \to f(x)$ in H. We assume without loss of generality that U is a compact neighborhood of x in X and that each $x_\lambda \in U$. Now $f[U]$ is compact in H^+, hence in H by the quoted theorem of Glicksberg [47, 1.2], so the net $f(x_\lambda)$ has a cluster point in $f[U] \subseteq H$. Since $f(x)$ is the only cluster point of $f(x_\lambda)$ in H^+, it is the only possible cluster point of $f(x_\lambda)$ in H. It follows that $f(x_\lambda)$ converges in $f[U] \subseteq H$ to its unique cluster point $f(x)$, as required. □

Corollary 12.3. *Let G and H be locally compact Abelian groups and let $f : G^+ \to H^+$ be a continuous function. Then f as a function from G to H is continuous.*

Proof. Surely $f : G \to H^+$ is continuous, so Lemma 12.2 applies. □

Discussion 12.4. Thus we see (cf. Trigos-Arrieta [107, 5.2], [105, 1.2]) that for locally compact Abelian groups and $h \in \mathrm{Hom}(G, H)$, the function $h : G \to H$ is continuous if and only if $h : G^+ \to H^+$ is continuous. This result is used in Chapter I of the thesis of Trigos-Arrieta [107] to study the relationship between the topologies of G and of G^+ for groups G in the class $\mathbb{LC}$ of locally compact Abelian groups and for groups which are k-spaces. The proof that Trigos-Arrieta gives for Corollary 12.3 does not make use of Glicksberg's Lemma 12.2, but relies upon the fact that the class $\mathbb{LC}$ respects compactness and *satisfies Pontrjagin duality*, i.e., the natural map $\upsilon : G \to \widehat{\widehat{G}}$ given by $\upsilon(x)(h) = h(x)$ for $x \in G$, $h \in \widehat{G}$ is a surjective homeomorphism (when $\widehat{G}$ and $\widehat{\widehat{G}}$ carry the compact-open topologies). We denote by $\mathbb{AG}$ the class of (Hausdorff) Abelian topological groups, and by $\mathbb{K}$ and $\mathbb{P}$ the subclasses of $\mathbb{AG}$ which respect compactness and satisfy Pontrjagin duality, respectively. We have, then, the (proper) class-theoretic containments $\mathbb{LC} \subset \mathbb{K} \cap \mathbb{P}$ and $\mathbb{LC} \subset \overset{\vee}{\mathbb{K}} \cap \mathbb{P}$, where $\overset{\vee}{\mathbb{K}}$ denotes the class $\overset{\vee}{\mathbb{K}} = \{\widehat{G} : G \in \mathbb{K}\}$. Trigos-Arrieta's approach to the study of G versus G^+ has as a consequence that the conclusion of Corollary 12.3

remains valid for groups G and H with $G \in \overset{\vee}{\mathbb{K}} \cap \mathbb{P}$ and $H \in \mathbb{P}$ (cf. in particular the proofs of Trigos-Arrieta [107, 5.2] and [105, 1.2]). I have been informed by Trigos-Arrieta that the assertion in [107, 6.3] and [105, 1.8] that the same conclusion holds for $G, H \in \mathbb{K} \cap \mathbb{P}$, using essentially the same proof, has in fact not been verified and remains open to question (see Question 12(b) below).

The foregoing paragraph suggests the following definition and questions, offered recently in correspondence by Trigos-Arrieta.

Notation 12.5. For $\mathbb{H} \subseteq \mathbb{AG}$ and for $G \in \mathbb{AG}$, write $G \in \mathcal{B}(\mathbb{H})$ (G belongs to the *Bohr hull of* $\mathbb{H}$) provided that each $h \in \mathrm{Hom}(G, H)$ with $H \in \mathbb{H}$ is continuous from G to H if and only if $h : G^+ \to H^+$ is continuous.

To fix ideas: In this notation the class $\mathbb{D}$ of discrete Abelian groups satisfies $\mathbb{D} \subseteq \mathcal{B}(\mathbb{H})$ for every $\mathbb{H} \subseteq \mathbb{AG}$; Corollary 12.4 gives the inclusion $\mathbb{LC} \subseteq \mathcal{B}(\mathbb{LC})$, and the extension of Corollary 12.4 cited above is the statement that

$$\overset{\vee}{\mathbb{K}} \cap \mathbb{P} \subseteq \mathcal{B}(\mathbb{P}).$$

In work not yet published answering in the negative a question posed in [107, 6.3] and [105, 1.8], Trigos-Arrieta has proved recently that the inclusion $\mathbb{K} \subseteq \mathcal{B}(\mathbb{K} \cap \mathbb{P})$ fails.

Question 12. Which (if any) of the inclusions
(a) $\mathbb{P} \subseteq \mathcal{B}(\mathbb{K} \cap \mathbb{P})$,
(b) $\mathbb{K} \cap \mathbb{P} \subseteq \mathcal{B}(\mathbb{K} \cap \mathbb{P})$,
(c) $\mathbb{P} \subseteq \mathcal{B}(\mathbb{P})$
are valid?

Remark 12.6. For more information on the class $\mathbb{K} \cap \mathbb{P}$, the reader is referred to Remus and Trigos-Arrieta [93, §2].

13. Haoxuan Zhou

Discussion 13.1. The paper of van Douwen [27] finds many zero-dimensional spaces X, all of them uncountable, such that no power X^κ is homogeneous. In what follows, we restructure Zhou's proof [113] that such a space may be chosen countable.

Theorem 13.2 (Zhou [113, 1.10]). *Let $p \in \omega^*$ and set $X = \omega \cup \{p\}$ in the topology inherited from $\beta(\omega)$. Then no power X^κ is homogeneous.*

Proof. The statement is obvious if $\kappa < \omega$, since in this case some points of X^κ are isolated and others are not. We assume in what follows that $\kappa \geqslant \omega$.

It is known that no sequence $I = \{n_k\colon k < \omega\}$ of integers satisfies $p = \lim_{k\to\infty} n_k$ [44, 14.25 and 14N.1]. (Indeed, if $I \in [\omega]^\omega$ and one writes $I = I_0 \cup I_1$ with I_0 and I_1 disjoint and infinite, then either $p \notin \bar{I_0}^{\beta(\omega)}$ or $p \notin \bar{I_1}^{\beta(\omega)}$.) The present proof unfolds upon the following lines. We define points $\underline{p}$ and $\underline{0}$ in X^κ by the rules

$$\underline{p}_\xi = p, \qquad \underline{0}_\xi = 0 \quad \text{for } \xi < \kappa,$$

and for $n < \omega$ we define $p(n) \in X^\kappa$ by

$$p(n)_\xi = \begin{cases} n & \text{if } \xi = 0, \\ p & \text{if } 0 < \xi < \kappa, \end{cases}$$

so that $\{p(n)\colon n < \omega\} \cup \{\underline{p}\} \simeq \omega \cup \{p\}$; then, given a (purported) surjective homeomorphism $h : X^\kappa \twoheadrightarrow X^\kappa$ with $h(\underline{p}) = \underline{0}$, we obtain the desired contradiction by showing that there is a sequence $I = \{n_k\colon k < \omega\}$ of integers such that $\underline{0} = \lim_{k\to\infty} h(p(n_k))$.

To show this, for $n < \omega$ define $V(n) \in \mathcal{N}(p(n))$ by

$$V(n) = \{n\} \times \prod_{0<\xi<\kappa} X_\xi = \bigl\{x \in X^\kappa\colon x_0 = n\bigr\},$$

let $W(n)$ be a basic neighborhood of $h(p(n))$ such that

$$h\bigl(p(n)\bigr) \in W(n) \subseteq h\bigl[V(n)\bigr],$$

define $C = \bigcup_{n<\omega} r(W(n))$ and write $[C]^{<\omega} = \{F(m)\colon m < \omega\}$. Now for $m < \omega$ define

$$U(m) = \pi^{-1}_{F(m)}\bigl(\pi_{F(m)}(\underline{0})\bigr) = \bigl\{x \in X^\kappa\colon \xi \in F(m) \Rightarrow x_\xi = 0\bigr\} \in \mathcal{N}(\underline{0})$$

and recall from [44, 6S.8] or from [20, 14.17(c), 15.18(b)] that

(∗) every non-empty G_δ-subset of ω^* has non-empty interior.

From (∗) and the homeomorphism $\{h(p(n))\colon n < \omega\} \cup \{\underline{0}\} \simeq \omega \cup \{p\}$ it follows that there is $I = \{n_k\colon k < \omega\} \in [\omega]^\omega$ such that

(∗∗) each $m < \omega$ satisfies $\bigl|\bigl\{k < \omega\colon h(p(n_k)) \notin U(m)\bigr\}\bigr| < \omega$.

To see that $h(p(n_k)) \to h(\underline{p}) = \underline{0}$, thus completing the proof, let U be a basic neighborhood of $\underline{0}$ in X^κ and find $m < \omega$ such that $r(U) \cap C = F(m)$. Now if $n < \omega$ satisfies $h(p(n)) \notin U$, then $p(n) \notin h^{-1}(U) \in \mathcal{N}(\underline{p})$ and hence $h^{-1}(U) \cap V(n) = \emptyset$, so $U \cap W(n) = \emptyset$; from $r(W(n)) \subseteq C$ then follows $U(m) \cap W(n) = \emptyset$, hence $h(p(n)) \notin U(m)$. From this analysis and (∗∗) it follows that

$$\bigl|\bigl\{k < \omega\colon h(p(n_k)) \notin U\bigr\}\bigr| < \omega \quad \text{for each } U \in \mathcal{N}(\underline{0})$$

so that $h(p(n_k)) \to h(\underline{p}) = \underline{0}$ as asserted. □

The thesis of Zhou [113] contains many contributions to the theory of homogeneity, especially to questions of this type: If X satisfies appropriate conditions, is X^ω necessarily homogeneous? (For related significant results in this direction, see Medvedev [73] and van Engelen [33].) Recently Dow and Pearl have proved the following result, which subsumes many of the theorems achieved by Zhou [113] and settles affirmatively a well-known, long-outstanding open question.

Theorem 13.3 (Dow and Pearl [31]). *Let X be a zero-dimensional, first countable space. Then X^ω is homogeneous.*

Discussion 13.4. It now appears that the principal remaining outstanding unsolved problem in this general area of investigation is the following. Here as usual we say that a space X is *strongly homogeneous* if X is homeomorphic to each of its non-empty open-and-closed subsets; it is well known and not difficult to prove that a strongly homogeneous first countable zero-dimensional space is homogeneous.

Questions 13.
(a) (Zhou [113, 1.9]). Let X be metrizable and zero-dimensional. Must X^ω be strongly homogeneous?
(b) What about the special case: X is separable?

It is noted by Zhou [113, 1.9] that there is an example of a (non-metrizable) first countable, zero-dimensional space X such that X^ω has $2^{\mathfrak{c}}$-many open subsets, each separable; evidently, X^ω is not strongly homogeneous.

14. Oscar E. Masaveu

Discussion 14.1. Following now-standard terminology introduced by Hewitt [54], one says that a space X is *resolvable* if X admits complementary dense subsets. Clearly a topological group with a proper dense subgroup is resolvable, but there are topological groups G with no proper dense subgroup (one may demand in addition, for example, that G be locally compact and Abelian [92] or totally bounded and Abelian [17]); thus the question of the resolvability of non-discrete topological groups is not totally trivial. Malykhin [71], working in the axiom system ZFC + P($\mathfrak{c}$), constructed on the Boolean group $\bigoplus_\omega \{0,1\}$ a group topology which is maximal among all non-discrete Hausdorff topologies; it is easy to see, and it was in fact noted decades earlier by Hewitt [54], that such a topology is irresolvable. Comfort and van Mill [18] showed (in ZFC) that Malykhin's example cannot be much generalized: If an Abelian group G has finite 2-rank (i.e., if G contains no copy of $\bigoplus_\omega \{0,1\}$) then G is resolvable in each of its non-discrete group topologies.

The theorem just cited asserts in effect that if in a non-discrete Abelian group G the subgroup $\mathcal{I}(G) := \{x \in G\colon\ x + x = 0\}$ is finite, then G is resolvable. Masaveu first generalized this result, replacing "finite" by "nowhere dense".

Lemma 14.2 (Masaveu [72, 2.3.1]; see also [16, 2.2]). *Every non-discrete Abelian group G such that $\mathcal{I}(G)$ is nowhere dense is resolvable.*

Corollary 14.3 (Masaveu [72, 3.2.1]; see also [16, 3.2]). *Every non-discrete irresolvable Abelian group G is of First Category in itself.*

Proof. It is enough to show that $\mathcal{I}(G)$, which by Discussion 14.1 is an open-and-closed subgroup, is of First Category in itself. To see this write $\mathcal{I}(G) = \bigoplus_{i\in I} \{0,1\}$, for $n < \omega$ set $D_n = \{x \in \mathcal{I}(G)\colon\ |\{i \in I\colon\ x_i \neq 0\}| > n\}$ and show (as in [72, §3.1] or [16, 3.1]) that the sets D_n are dense and open in $\mathcal{I}(G)$ and satisfy $\bigcap_n D_n = \emptyset$. □

The following lemma, discovered at Wesleyan in ignorance of an earlier paper of Malykhin [70], generalizes a result given there (the case $\kappa = \omega$).

Theorem 14.4 (Masaveu [72, 4.1.1]; see also [16, 4.1]). *Let $\kappa \geqslant \omega$. Let X and Y be spaces with strictly increasing families $\{X_\eta\colon \eta < \kappa\}$ and $\{Y_\eta\colon \eta < \kappa\}$ such that*

(i) $X = \bigcup_{\eta<\kappa} X_\eta$ *and* $Y = \bigcup_{\eta<\kappa} Y_\eta$, *and*

(ii) *each η satisfies* $\mathrm{int}_X X_\eta = \mathrm{int}_Y Y_\eta = \emptyset$.

Then $X \times Y$ is resolvable.

Theorem 14.4 has the following curious consequence.

Theorem 14.5 (Masaveu [72, 4.2.1]; see also [16, 4.5]). *Let G, G_1 and G_2 be non-discrete topological groups with G_1 and G_2 Abelian. Then $G \times G$ and $G_1 \times G_2$ are resolvable.*

Proof. The statement concerning $G \times G$ follows easily from Theorem 14.4. The statement concerning $G_1 \times G_2$ follows from Corollary 14.3 if either group is not of First Category in itself, so we write $G_i = \bigcup_{n<\omega} F_{i,n}$ with each $F_{i,n}$ nowhere dense in G_i. Then $G_1 \times G_2$ is resolvable by Corollary 14.3. □

There is given by Malykhin [70] in a suitable model of ZFC a T_1-space X without isolated points such that $X \times X$ is irresolvable. This and Theorem 14.5 leave unresolved the following questions about resolvability in product spaces.

Questions 14 (Masaveu [72]; see also [16]). Are there examples, either consistent or absolute, of Hausdorff spaces (Tychonoff spaces?) X_0 and X_1 without isolated points such that $X_0 \times X_1$ is not resolvable? If so, can the spaces X_i be chosen homogeneous? To be topological groups?

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